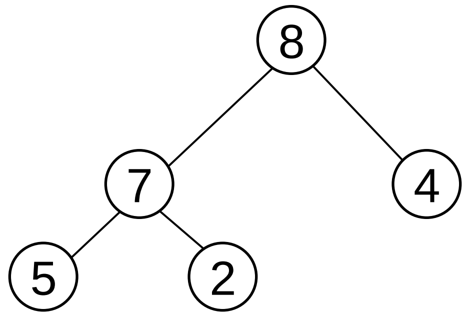
The Heap Data Structure

- *Def:* A **heap** is a nearly complete binary tree with the following two properties:
  - **Structural property:** all levels are full, except possibly the last one, which is filled from left to right
  - **Order (heap) property:** for any node x

$$\text{Parent}(x) \geq x$$



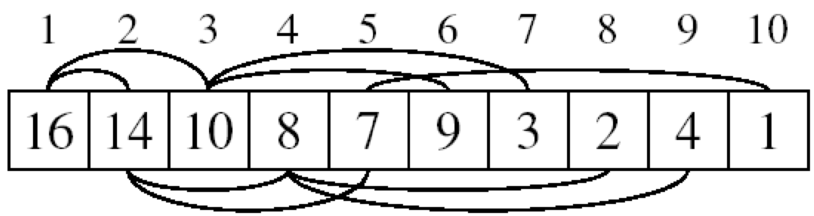
Heap

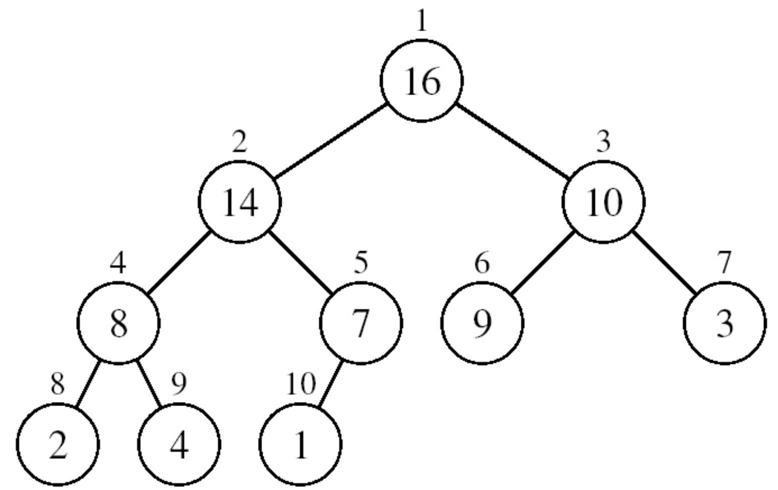
From the heap property, it follows that:
"The root is the maximum element of the heap!"

A heap is a binary tree that is filled in order

Array Representation of Heaps

- A heap can be stored as an array *A*.
  - Root of tree is $A[1]$
  - Left child of $A[i] = A[2i]$
  - Right child of $A[i] = A[2i + 1]$
  - Parent of $A[i] = A[\lfloor i/2 \rfloor]$
  - Heapsize[A] ≤ length[A]
- The elements in the subarray $A[(\lfloor n/2 \rfloor+1) .. n]$ are leaves





Heap Types

- **Max-heaps** (largest element at root), have the *max-heap property:*
  - for all nodes i, excluding the root:

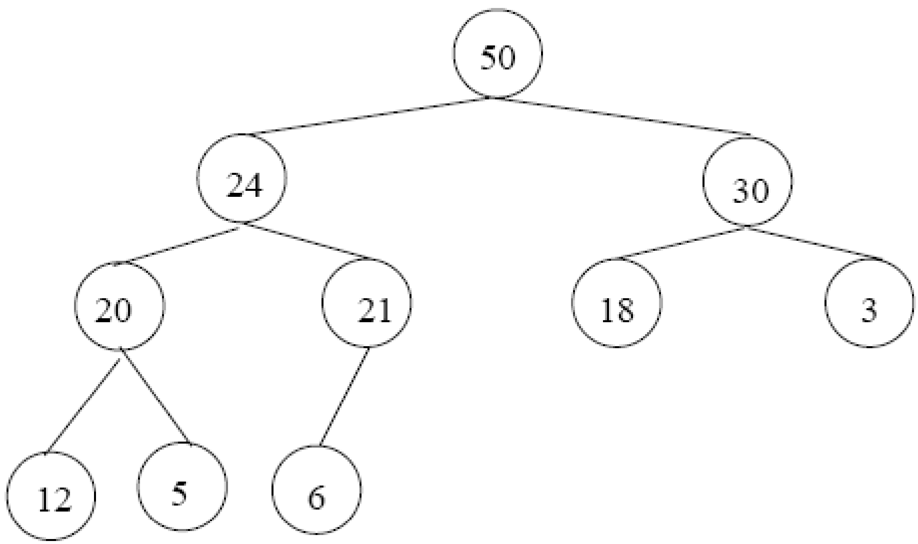
$$A[\text{PARENT}(i)] \geq A[i]$$

- **Min-heaps** (smallest element at root), have the *min-heap property:*
  - for all nodes i, excluding the root:

$$A[\text{PARENT}(i)] \leq A[i]$$

Adding/Deleting Nodes

- New nodes are always inserted at the bottom level (left to right)
- Nodes are removed from the bottom level (right to left)

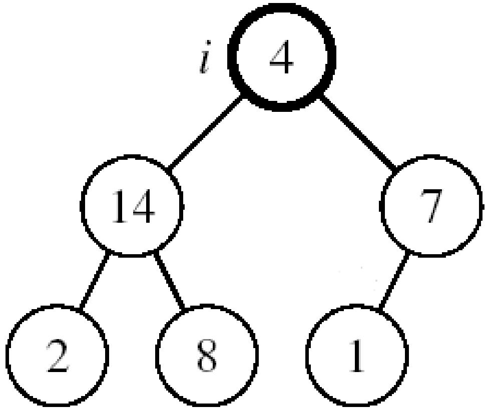


Operations on Heaps

- Maintain/Restore the max-heap property
  - MAX-HEAPIFY
- Create a max-heap from an unordered array
  - BUILD-MAX-HEAP
- Sort an array in place
  - HEAPSORT
- Priority queues

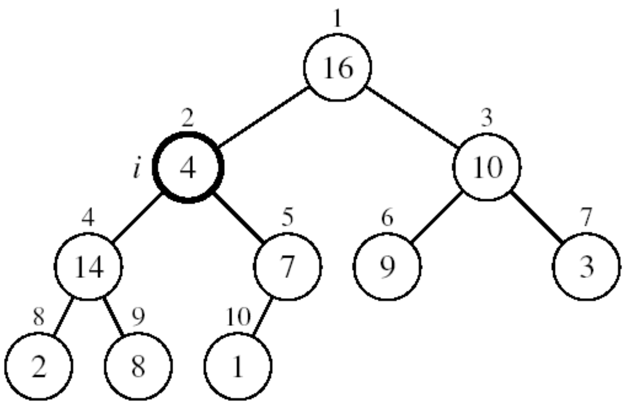
Maintaining the Heap Property

- Suppose a node is smaller than a child
  – Left and Right subtrees of *i* are max-heaps
- To eliminate the violation:
  – Exchange with larger child
  – Move down the tree
  – Continue until node is not smaller than children



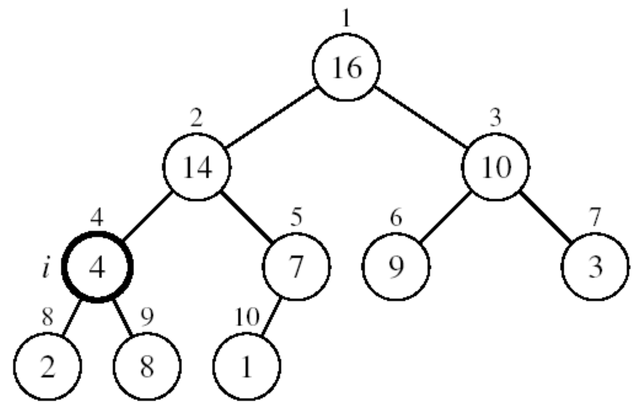
Example

MAX-HEAPIFY(A, 2, 10)



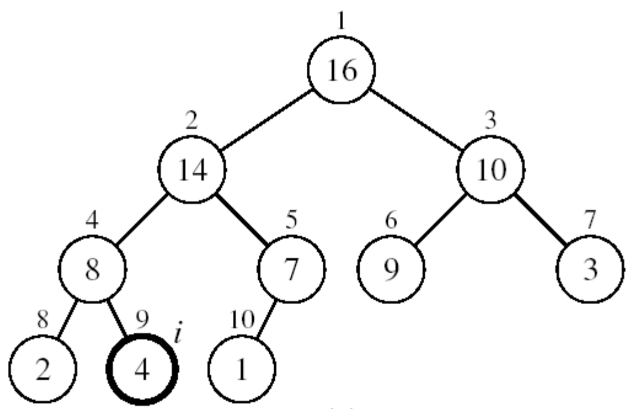
A[2] violates the heap property

A[2] ↔ A[4]



A[4] violates the heap property

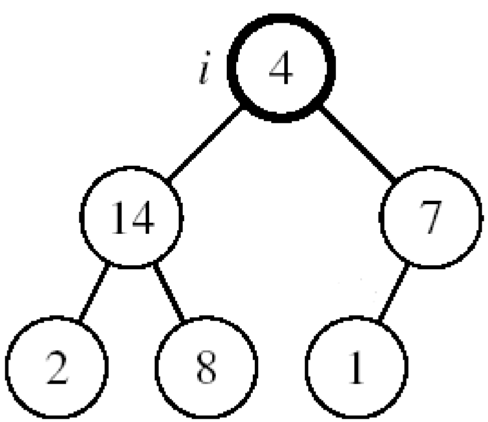
A[4] ↔ A[9]



Heap property restored

Maintaining the Heap Property

- Assumptions:
  - Left and Right subtrees of i are max-heaps
  - $A[i]$ may be smaller than its children



Alg: MAX-HEAPIFY(A, i, n)

$l \leftarrow$ LEFT(i)
$r \leftarrow$ RIGHT(i)
if $l \leq n$ and $A[l] > A[i]$
    then $largest \leftarrow l$
    else $largest \leftarrow i$
if $r \leq n$ and $A[r] > A[largest]$
    then $largest \leftarrow r$
if $largest \neq i$
    then exchange $A[i] \leftrightarrow A[largest]$
        MAX-HEAPIFY(A, $largest$, n)

MAX-HEAPIFY Running Time

- Intuitively:

  - It traces a path from the root to a leaf (longest path length: h)
  - At each level, it makes exactly 2 comparisons
  - Total number of comparisons is 2h
  - Running time is O(h) or $O(lgn)$

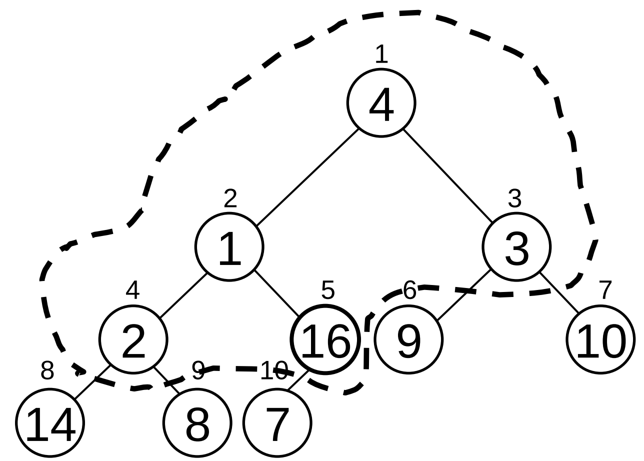
- Running time of MAX-HEAPIFY is $O(lgn)$
- Can be written in terms of the height of the heap, as being $O(h)$
  - Since the height of the heap is $\lfloor lgn \rfloor$

Building a Heap

- Convert an array $A[1 \ldots n]$ into a max-heap ($n = length[A]$)
- The elements in the subarray $A[(\lfloor n/2 \rfloor+1) \ldots n]$ are leaves
- Apply MAX-HEAPIFY on elements between 1 and $\lfloor n/2 \rfloor$

Alg: <u>BUILD-MAX-HEAP(A)</u>

```
n = length[A]
  for i ← ⌊n/2⌋ downto 1
       do MAX-HEAPIFY(A, i, n)
```

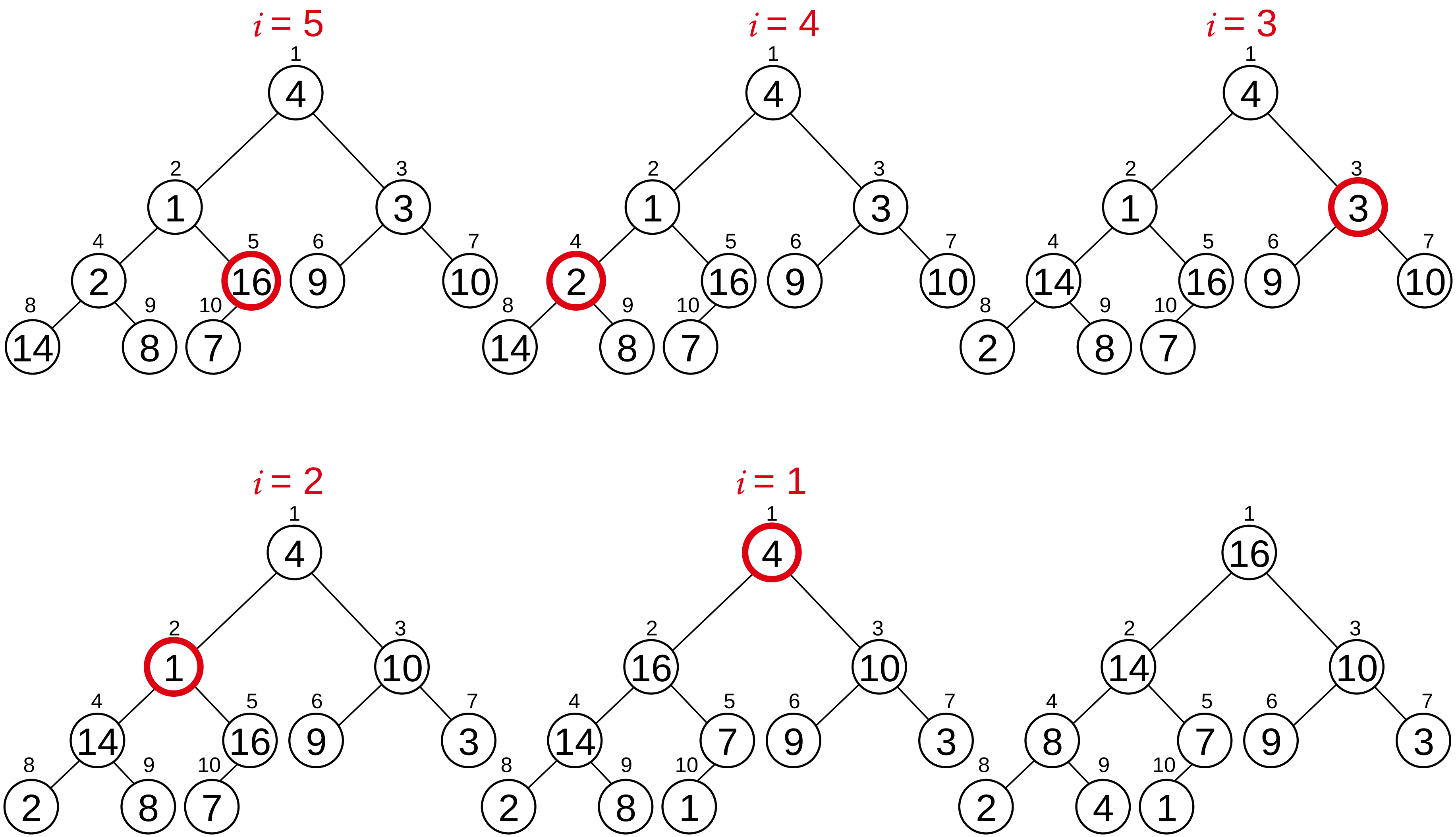


A:

4	1	3	2	16	9	10	14	8	7

Example:

A										
	4	1	3	2	16	9	10	14	8	7



Running Time of BUILD MAX HEAP

Alg: BUILD-MAX-HEAP(A)

1. n = length[A]
2. **for** $i \leftarrow \lfloor n/2 \rfloor$ **downto** 1
3. **do** MAX-HEAPIFY(A, i, n)

$$\left.\begin{array}{l} \text{2. for } i \leftarrow \lfloor n/2 \rfloor \text{ downto } 1 \\ \text{3. do MAX-HEAPIFY}(A, i, n) \quad O(\lg n) \end{array}\right\} O(n)$$

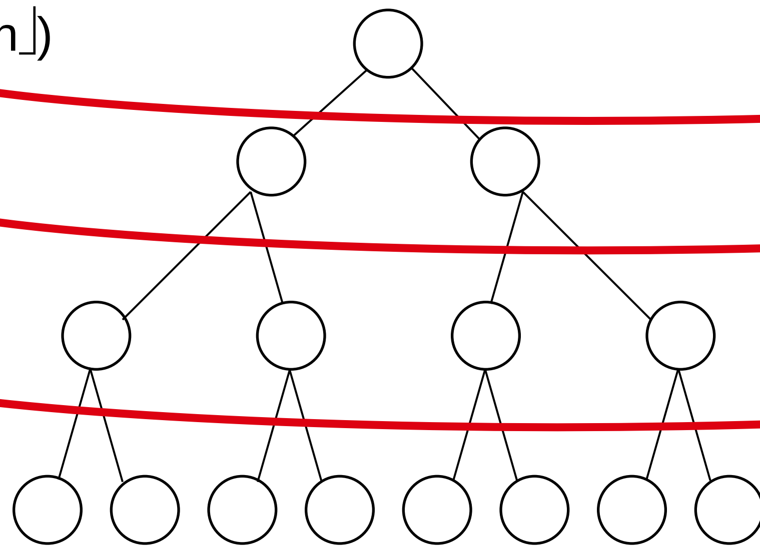
$\Rightarrow$ Running time: $O(n\lg n)$

- This is not an asymptotically tight upper bound

Running Time of BUILD MAX HEAP

- HEAPIFY takes $O(h)$ $\Rightarrow$ the cost of HEAPIFY on a node i is proportional to the height of the node i in the tree

$$\Rightarrow T(n) = \sum_{i=0}^{h} n_i h_i = \sum_{i=0}^{h} 2^i (h-i) = O(n)$$



Height	Level	No. of nodes
$h_0 = 3$ ($\lfloor \lg n \rfloor$)	$i = 0$	2^0
$h_1 = 2$	$i = 1$	2^1
$h_2 = 1$	$i = 2$	2^2
$h_3 = 0$	$i = 3$ ($\lfloor \lg n \rfloor$)	2^3

$h_i = h - i$ height of the heap rooted at level i

$n_i = 2^i$ number of nodes at level i

Running Time of BUILD MAX HEAP

$$T(n) = \sum_{i=0}^{h} n_i h_i$$ Cost of HEAPIFY at level i $*$ number of nodes at that level

$$= \sum_{i=0}^{h} 2^i (h-i)$$ Replace the values of n_i and h_i computed before

$$= \sum_{i=0}^{h} \frac{h-i}{2^{h-i}} 2^h$$ Multiply by 2^h both at the nominator and denominator and write 2^i as $\frac{1}{2^{-i}}$

$$= 2^h \sum_{k=0}^{h} \frac{k}{2^k}$$ Change variables: k = h - i

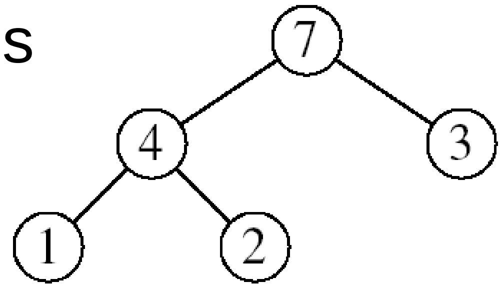
$$\le n \sum_{k=0}^{\infty} \frac{k}{2^k}$$ The sum above is smaller than the sum of all elements to ∞ and h = lgn

$$= O(n)$$ The sum above is smaller than 2

Running time of BUILD-MAX-HEAP: $T(n) = O(n)$

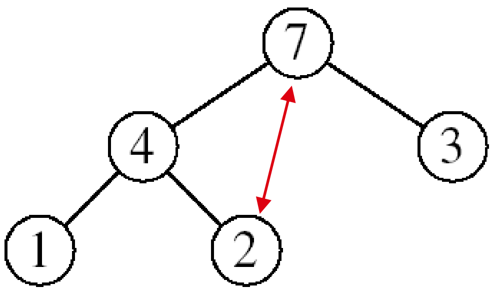
Heapsort

- Goal:
  - Sort an array using heap representations

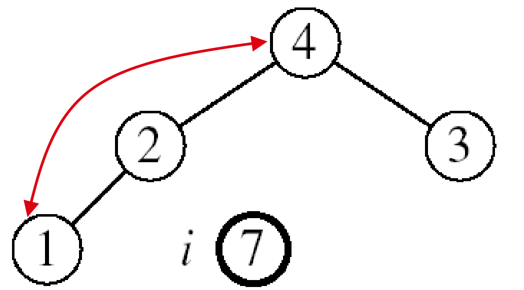


- Idea:
  - Build a **max-heap** from the array
  - Swap the root (the maximum element) with the last element in the array
  - "Discard" this last node by decreasing the heap size
  - Call MAX-HEAPIFY on the new root
  - Repeat this process until only one node remains

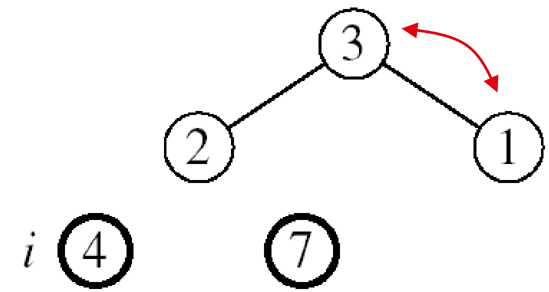
Example: A=[7, 4, 3, 1, 2]



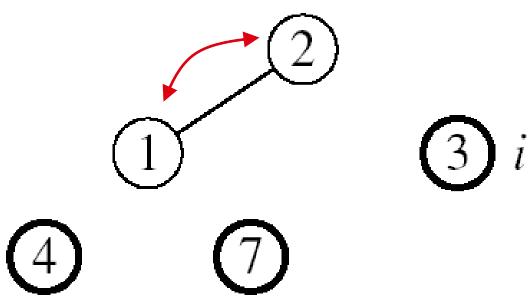
MAX-HEAPIFY(A, 1, 4)



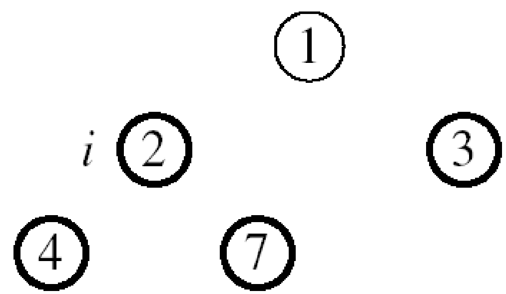
MAX-HEAPIFY(A, 1, 3)

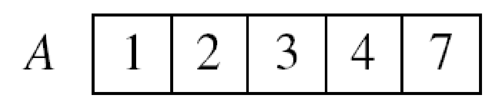


MAX-HEAPIFY(A, 1, 2)



MAX-HEAPIFY(A, 1, 1)





Alg: HEAPSORT(*A*)

1. BUILD-MAX-HEAP(A) $O(n)$
2. **for** $i \leftarrow length[A]$ **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. MAX-HEAPIFY(A, 1, i - 1) $O(\lg n)$

$n-1$ times

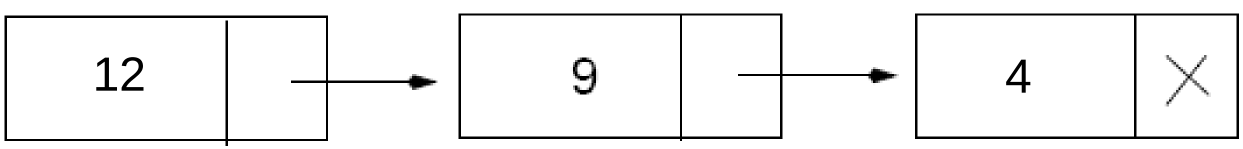
- Running time: $O(n\lg n)$ --- Can be shown to be $\Theta(n\lg n)$

Priority Queues

Properties

- Each element is associated with a value (priority)

- The key with the highest (or lowest) priority is extracted first



Operations on Priority Queues

- Max-priority queues support the following operations:
  - INSERT(S, x): inserts element x into set S
  - EXTRACT-MAX(S): removes and returns element of S with largest key
  - MAXIMUM(S): returns element of S with largest key
  - INCREASE-KEY(S, x, k): increases value of element x's key to k (Assume $k \geq x$'s current key value)

HEAP-MAXIMUM

Goal:

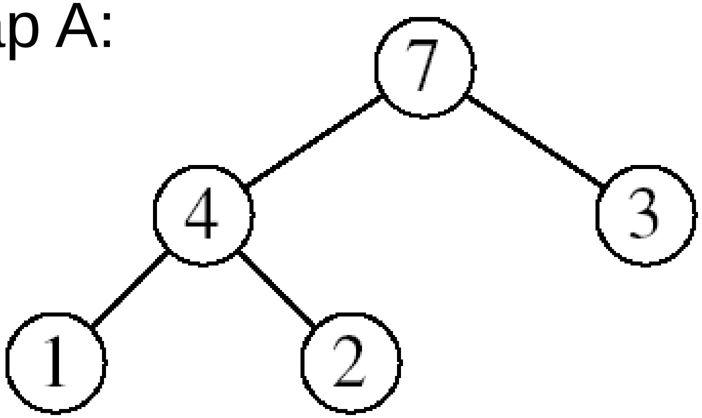
– Return the largest element of the heap

Running time: $O(1)$

Alg: HEAP-MAXIMUM(*A*)

1. **return** $A[1]$

Heap A:



Heap-Maximum(A) returns 7

HEAP-EXTRACT-MAX

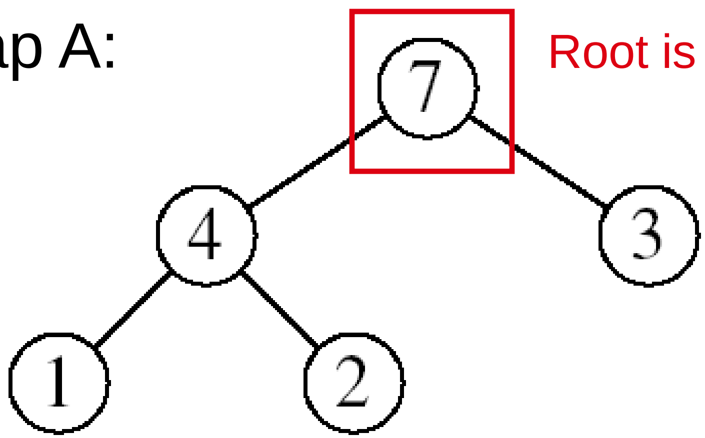
Goal:

– Extract the largest element of the heap (i.e., return the max value and also remove that element from the heap

Idea:

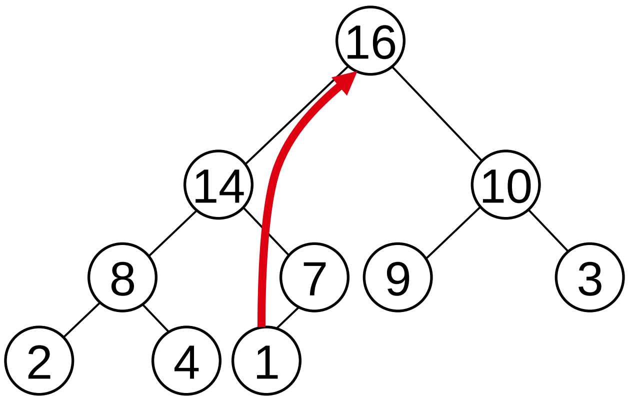
– Exchange the root element with the last
– Decrease the size of the heap by 1 element
– Call MAX-HEAPIFY on the new root, on a heap of size n-1

Heap A:

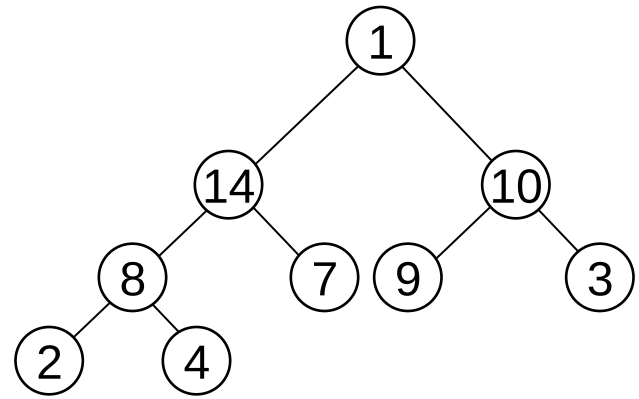


Root is the largest element

Example: HEAP-EXTRACT-MAX

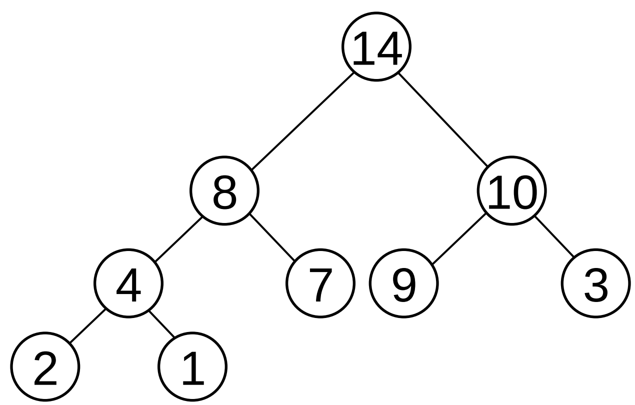


max = 16



Heap size decreased with 1

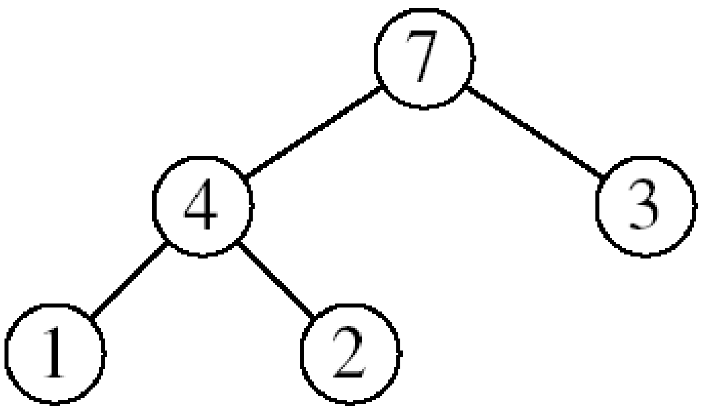
Call MAX-HEAPIFY(A, 1, n-1)



HEAP-EXTRACT-MAX

Alg: HEAP-EXTRACT-MAX(A, n)

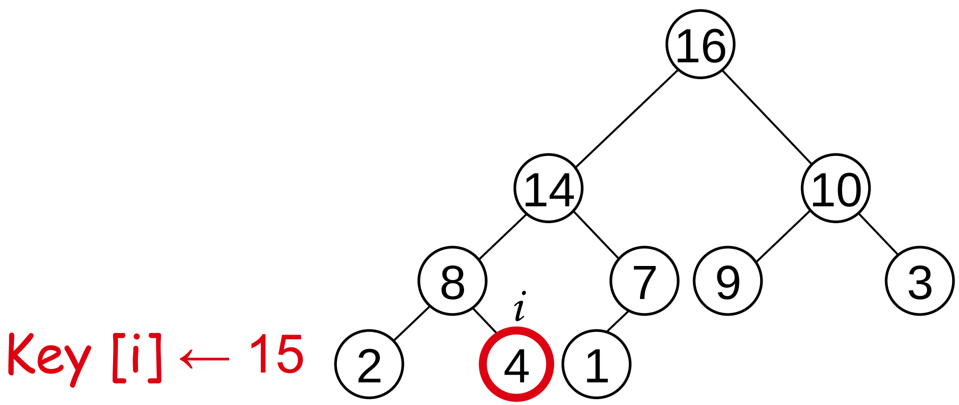
1. **if** $n < 1$
2. **then error** "heap underflow"
3. $max \leftarrow A[1]$
4. $A[1] \leftarrow A[n]$
5. MAX-HEAPIFY(A, 1, n-1) ▷ remakes heap
6. **return** max



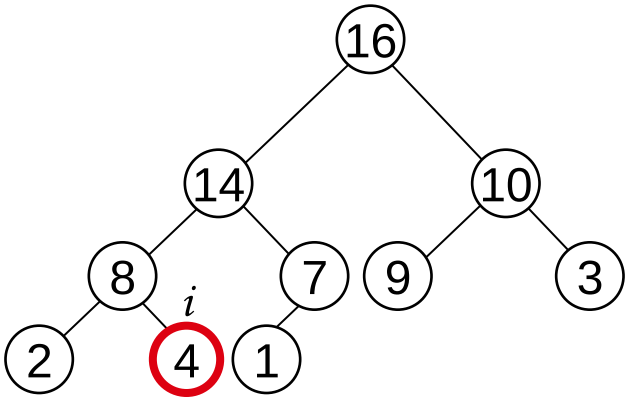
Running time: $O(\lg n)$

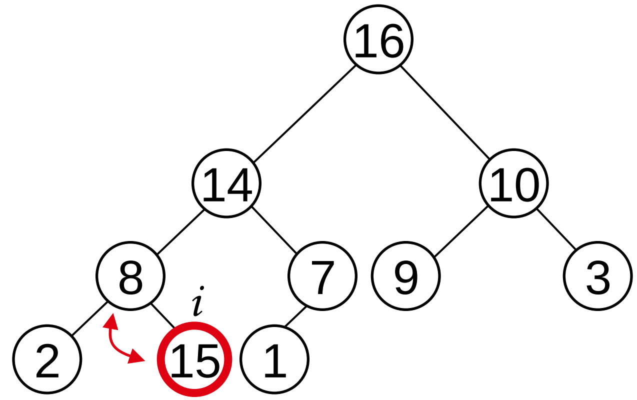
HEAP-INCREASE-KEY

- Goal:
  – Increases the key of an element i in the heap
- Idea:
  – Increment the key of $A[i]$ to its new value
  – If the max-heap property does not hold anymore: traverse a path toward the root to find the proper place for the newly increased key

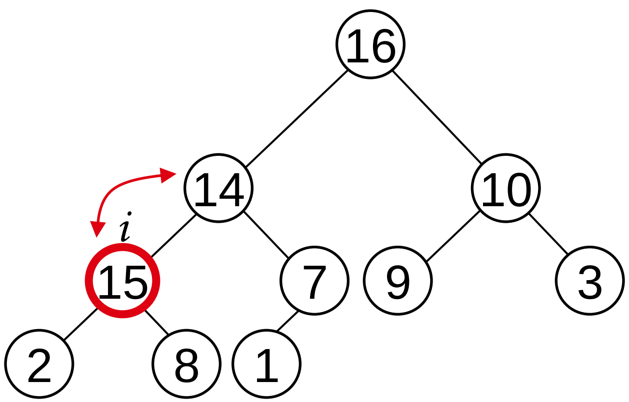


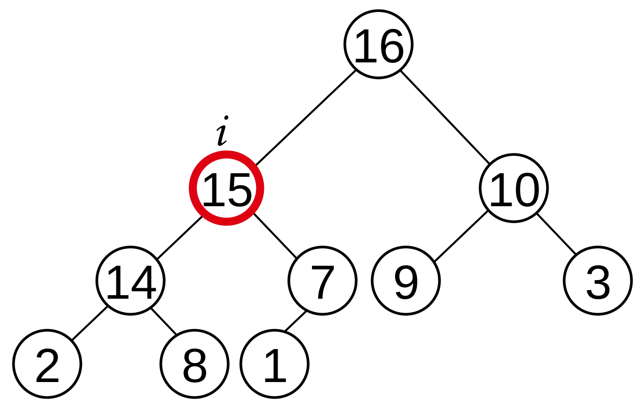
Example: HEAP-INCREASE-KEY





$Key[i] \leftarrow 15$



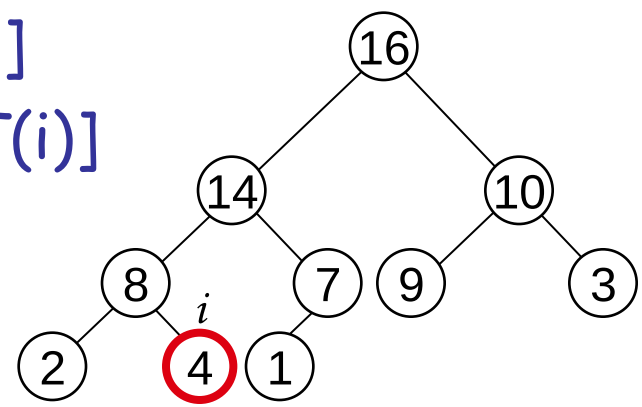


HEAP-INCREASE-KEY

Alg: HEAP-INCREASE-KEY(*A*, i, key)

```
if key < A[i]
   then error "new key is smaller than current key"
A[i] ← key
while i > 1 and A[PARENT(i)] < A[i]
    do exchange A[i] ↔ A[PARENT(i)]
        i ← PARENT(i)
```

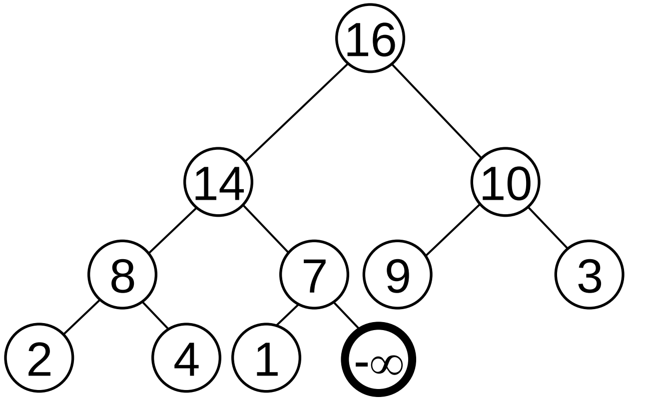
- Running time: $O(\lg n)$

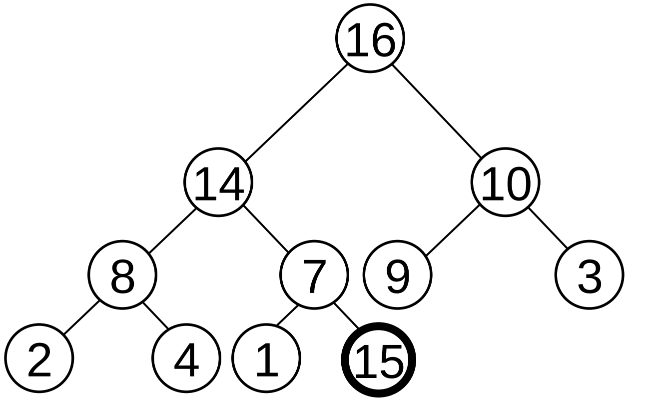


Key [i] ← 15

MAX-HEAP-INSERT

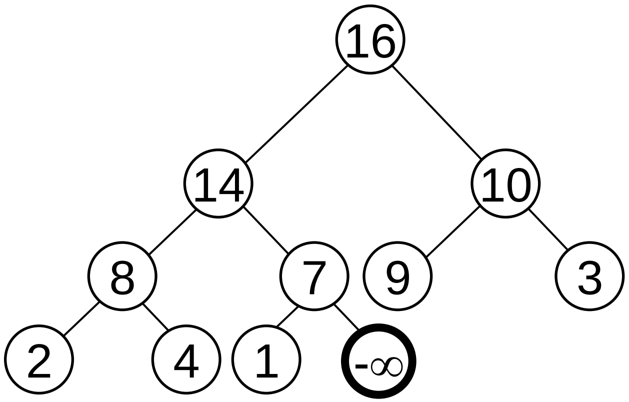
- Goal:
  - Inserts a new element into a max-heap
- Idea:
  - Expand the max-heap with a new element whose key is $-\infty$
  - Calls HEAP-INCREASE-KEY to set the key of the new node to its correct value and maintain the max-heap property



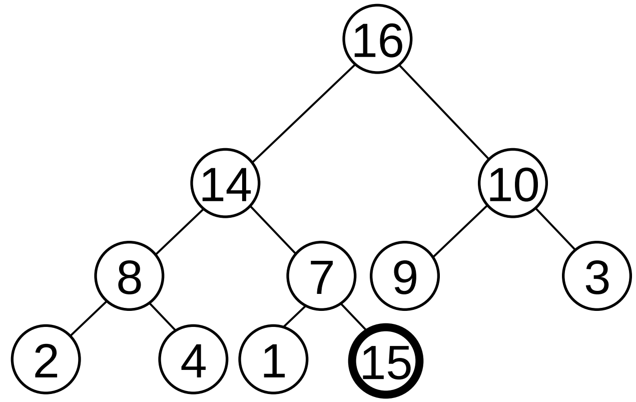


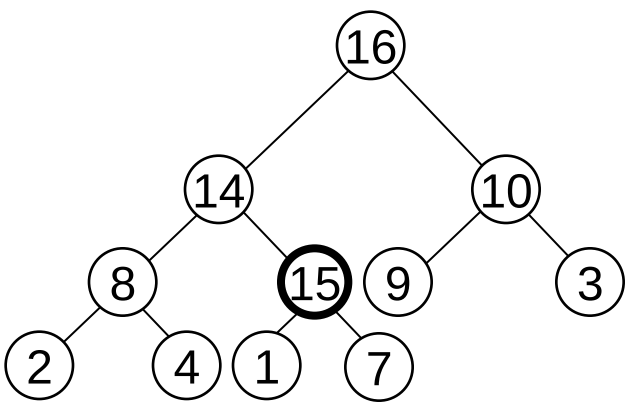
Example: MAX-HEAP-INSERT

Insert value 15:
- Start by inserting -∞

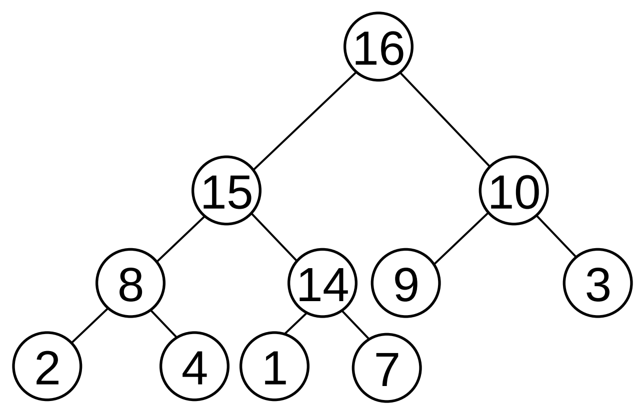


Increase the key to 15
Call HEAP-INCREASE-KEY on A[11] = 15

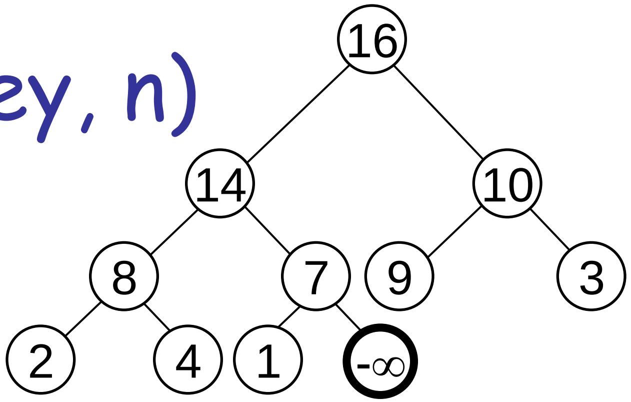




The restored heap containing the newly added element



MAX-HEAP-INSERT



Alg: MAX-HEAP-INSERT(A, key, n)

1. heap-size[A] $\leftarrow n + 1$
2. $A[n + 1] \leftarrow -\infty$
3. HEAP-INCREASE-KEY(A, $n + 1$, key)

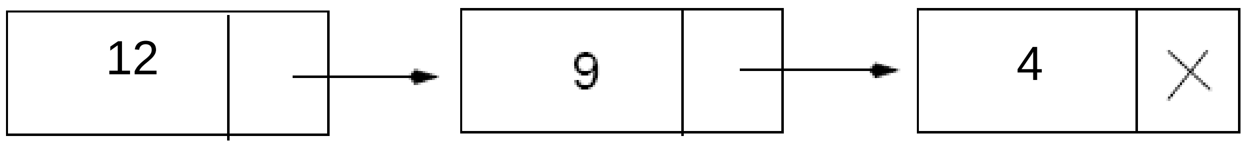
Running time: $O(\lg n)$

Summary

- We can perform the following operations on heaps:
  - MAX-HEAPIFY $O(\lg n)$
  - BUILD-MAX-HEAP $O(n)$
  - HEAP-SORT $O(n\lg n)$
  - MAX-HEAP-INSERT $O(\lg n)$
  - HEAP-EXTRACT-MAX $O(\lg n)$
  - HEAP-INCREASE-KEY $O(\lg n)$
  - HEAP-MAXIMUM $O(1)$

Average $O(\lg n)$

Priority Queue Using Linked List



Remove a key: O(1)

Insert a key: O(n)

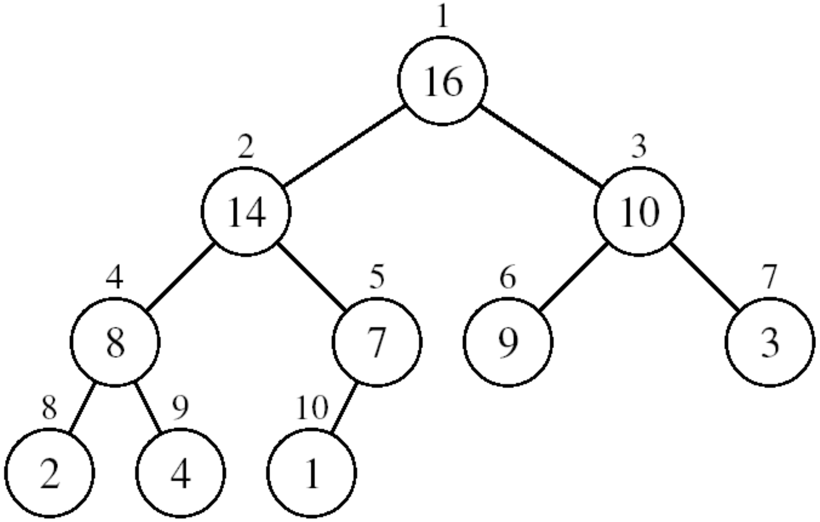
Increase key: O(n)

Extract max key: O(1)

Average: O(n)

Problems

Assuming the data in a max-heap are distinct, what are the possible locations of the second-largest element?



Problems

(a) What is the maximum number of nodes in a max heap of height h?

(b) What is the maximum number of leaves?

(c) What is the maximum number of internal nodes?

Problems

- Demonstrate, step by step, the operation of Build-Heap on the array

  A=[5, 3, 17, 10, 84, 19, 6, 22, 9]

Problems

- Let A be a heap of size n. Give the most efficient algorithm for the following tasks:

(a) Find the sum of all elements

(b) Find the sum of the largest lgn elements